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A NUMERICAL SCHEME FOR STEADY. SUPERCRITICAL FLOWS WITH BOUNDARY FITTED COORDINATES

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ABSTRACT

An algorithm based on flux difference splitting is presented for the solution of two-dimensional, steady, supercritical open channel flows. A transformation maps a non-rectangular, physical domain into a rectangular one. The governing equations are then the shallow water equations, including terms of slope and friction, in a generalised coordinate system. A regular mesh on a rectangular computational domain can then be employed. The resulting scheme has good jump capturing properties and the advantage of using boundary/body-fitted meshes. The scheme is applied to a problem of flow in a river whose geometry induces a region of supercritical flow.

KEY WORDS Flux difference splitting Supercritical flow Open channel flow Transformation mapping

LIST OF SYMBOLS

F, G	flux functions
f	source term
w	Cartesian coordinates
ξ, η	generalized coordinates
g	acceleration due to gravity
ϕ	$g \times$ total height
u, w	component of fluid velocity in x, z
s	source from slope of energy grade lines
h	undisturbed depth
U, W	transformed velocities
J	Jacobian of transformation
n	Manning's roughness coefficient
A	Jacobian of flux functions
λ_i	eigenvalues
e_i, r_i	eigenvectors
$\tilde{\phi}$, $\tilde{u}$, $\tilde{w}$, X, Z, X', Z'	averages
$\tilde{\lambda}_i$	approximate eigenvalues
$\tilde{e}_i$, $\tilde{r}_i$	approximate eigenvectors

INTRODUCTION

Hydraulic engineering applications frequently deal with open channels carrying supercritical flows. Typical examples are sewer systems, spillways, and naturally occurring mountainous

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streams and rivers during heavy rainfall. It is widely accepted that the analysis and design of channels having supercritical flow can be carried out using this model[1,2,3,4,5,6,7,8,9,10]

In this paper we present an approximate Riemann solver for the equations governing steady, two-dimensional, supercritical open channel flows, including slope and friction terms. The scheme incorporates generalised coordinates so that a non-rectangular physical domain can be mapped on to a rectangular computational domain, and a boundary fitted mesh can be employed. The numerical results are included for a test problem.

GOVERNING EQUATIONS

The steady, two-dimensional equations governing supercritical open channel flow can be written in terms of generalised coordinates ζ, η as,

$$F_\zeta + G_\eta = f \tag{1}$$

where

$$F(w) = (\phi U, z_\eta \phi^2/_2 + \phi u U, -x_\eta \phi^2/_2 + \phi w U)^T \tag{2a}$$

$$G(w) = (\phi W, -z_\zeta \phi^2/_2 + \phi u W, x_\zeta \phi^2/_2 + \phi w W)^T \tag{2b}$$

$$f(w) = (0, g\phi(z_\eta h_\zeta - z_\zeta h_\eta - s(x_\eta W + x_\zeta U)), g\phi(x_\zeta h_\eta - x_\eta h_\zeta - s(z_\eta W + z_\eta U)))^T \tag{2c}$$

and

$$w = (\phi,\ \phi u,\ \phi w)^T, \tag{2d}$$

where the transformation from physical (x, z) space to computational (ζ, η) space is known, either as $x = x(\zeta, \eta)$, $z = z(\zeta, \eta)$, or $\zeta = \zeta(x, z)$, $\eta = \eta(x, z)$. This is discussed in more detail later. The quantities $(\phi, u, w)(\zeta, \eta)$ represent g multiplied by the total height above the bottom of the channel and the components of the fluid velocity in the x and z directions, respectively. The gravitational constant is g and the undisturbed depth of the water is given by $h(x, z) = h(x(\zeta, \eta), z(\zeta, \eta))$. Also,

$$s = \frac{n^2\sqrt{u^2 + w^2}}{(\phi/g)^{4/3}} \tag{3}$$

as determined from the slopes of the energy grade lines, and n is Manning's roughness coefficient. Finally,

$$U = z_\eta u - x_\eta w, \qquad W = -z_\zeta u + x_\zeta w. \tag{4a–d}$$

Note that

$$(\zeta_x, \zeta_z, \eta_x, \eta_z) = \frac{1}{J} z_\eta, -x_\eta, -z_\zeta, x_\zeta) \tag{5a}$$

where

$$J = x_\zeta z_\eta - x_\eta z_\zeta = \frac{1}{\zeta_x \eta_z - \zeta_z \eta_x} \tag{5b}$$

is the Jacobian of the transformation.

We consider first the approximate solution of (1) in the case $f = 0$, and then modify the scheme to treat the case when $f \neq 0$.

SPACE MARCHING

Consider solving (1) in the case $f = 0$ with F, G as defined by (2a–b), which can be written as,

$$F_\zeta + AF_\eta = 0. \tag{6}$$

If the Jacobian $A = \partial G/\partial F$ has real, distinct eigenvalues then the system (6) is hyperbolic and we shall assume this to be the case here. This assumption corresponds to the flow governed by (6) being supercritical everywhere. Thus, it is appropriate to use techniques similar to those developed for time dependent conservation laws of the form,

$$c_t + H(c)_x = 0, \qquad c(x, 0) = c_0(x) \tag{7}$$

i.e.

$$c_t + Ec_x = 0 \tag{8}$$

where c is the conserved variable and $E = \partial H/\partial c$. Instead of marching forward in time 't', for (4a–b) we march forward in the space variable 'ζ', for example. In particular, smooth solutions of (7) will develop discontinuities (shock) in time and likewise (6) will exhibit oblique jumps in space.

A specific technique for solving (7) in the case of the unsteady Euler equations with real gases was given in Reference 11 and we develop similar ideas here for solving (6). The scheme in Reference 11 represents an approximation to the scheme of Godunov[12], and the scheme presented here can be viewed in a similar way. Although we use the concept of a Riemann problem as in Reference 11 by identifying 'η' as a time like variable, three important differences remain. Firstly, the structure differs as a result of the equations of flow being different, and the resulting construction of the scheme is more detailed, although its actual implementation is just as straightforward. Secondly, the scheme applies only to flows that are wholly supercritical in at least one direction, which we take, without loss of generality, to be the ζ-direction; and finally, the flow is assumed to be steady. Thus it is imperative that the ζ-axis is aligned with one of these directions, e.g. the predominant flow direction.

LINEARISED RIEMANN PROBLEM

If an approximate solution of equations (6) is sought (along a η-coordinate line $\zeta = \zeta_j$) using a finite difference method then the solution is known at a set of discrete mesh points $(\zeta, \eta) = (\zeta_j, \eta_k)$. The approximate solution w_k^j for $\eta \in (\eta_k - \Delta\eta/2, \eta_k + \Delta\eta/2)$ where $\Delta\eta = \eta_k - \eta_{k-1}$ is a constant mesh spacing (see e.g. Reference 12). A Riemann problem is now present at each interface $\eta_{k-1/2} = \frac{1}{2}(\eta_{k-1} + \eta_k)$ separating adjacent states w_{k-1}^j, w_k^j. If the steady equations (6) are linearised by considering the Jacobian matrix $A = \partial G/\partial F$ to be constant in each interval (η_{k-1}, η_k), say $\tilde{A} = \tilde{A}(w_{k-1}^j, w_k^j)$, then the resulting equations

$$F_\zeta + \tilde{A}F_\eta = O \tag{9}$$

can be solved approximately using explicit space marching in the ζ-direction. The step $\Delta\zeta$ is restricted so that the solutions of adjacent Riemann problems do not interact, and this sets a restriction on the mesh spacing in the ζ-direction once $\Delta\eta$ has been chosen. The scalar problems that result from this analysis can be solved by upwind differencing consistent with the theory of characteristics; however, an approximate Jacobian matrix needs to be constructed in each interval so that jumps can be captured automatically.

NUMERICAL SCHEME

Consider (6) and begin by noting the structure associated with (6).

Structure

Let $B = \partial F/\partial w$ and $C = \partial G/\partial w$ denote the Jacobians of the flux functions F and G, so that the Jacobian $A = \partial G/\partial F = CB^{-1}$. Denote also the eigenvalues and eigenvectors of A by λ_i, $r_i = 1, 2, 3$, respectively, so that,

$$Ar_i = \lambda_i r_i, \qquad i = 1, 2, 3 \tag{10}$$

and hence

$$(C-\lambda_i B)e_i=0, \qquad i=1, 2, 3 \tag{11}$$

where

$$Be_i=r_i, \qquad i=1, 2, 3 \tag{12}$$

Another associated matrix is $\mathscr{A}=B^{-1}C$ since (4a) can be written as,

$$w_\zeta+\mathscr{A}w_\eta=O \tag{13}$$

and from (5.2)

$$(B^{-1}C-\lambda_i I)e_i=O \qquad i=1, 2, 3 \tag{14}$$

so that $\mathscr{A}$ has eigenvalues λ_i with eigenvectors e_i.

The matrices B and C are,

$$B=\begin{bmatrix} 0 & z_\eta & -x_\eta \\ z_\eta(\phi-u^2)+uwx_\eta & 2uz_\eta-wx_\eta & -ux_\eta \\ -x_\eta(\phi-w^2)-uwz_\eta & wz_\eta & uz_\eta-2wx_\eta \end{bmatrix} \tag{15}$$

and

$$C=\begin{bmatrix} 0 & -z_\zeta & x_\zeta \\ -z_\zeta(\phi-u^2) & -2uz_\zeta+wx_\zeta & ux_\zeta \\ x_\zeta(\phi-w^2)+uwz_\zeta & -wz_\zeta & 2wx_\zeta-uz_\zeta \end{bmatrix} \tag{16}$$

Solving, we find that,

$$\lambda_{1,2,3}=\frac{\mu x_\zeta-z_\zeta}{z_\eta-\mu x_\eta} \tag{17a–c))}$$

where μ satisfies,

$$(u^2-\phi)\mu^2-2uw\mu+w^2-\phi=0 \tag{18a}$$

or

$$\mu=\frac{w}{u}, \tag{18b}$$

so

$$\lambda_{1,2}=\frac{UW+\phi(x_\zeta x_\eta+z_\zeta z_\eta)\pm\phi\sqrt{d}}{U^2-\phi(x_\eta^2+z_\eta^2)} \tag{19a–b}$$

$$\lambda_3=\frac{W}{U}, \tag{19c}$$

where

$$d=\frac{(U^2(x_\zeta^2+z_\zeta^2)+W^2(x_\eta^2+z_\eta^2))}{\phi}-J^2 \tag{20}$$

and J is the Jacobian in (5b).

Construction of $\tilde{A}$

In constructing numerical solutions to (6) it is our aim, as stated previously, to obtain an approximation to the Jacobian $A=\partial G/\partial F$ in an interval (η_{k-1}, η_k), so that approximate solutions can be sought to the linearized Riemann problem (9).

Consider two adjacent states $w^j_{k-1}=w_L$ and $w^j_k=w_R$ (left and right) given at either end of the cell $(\eta_{k-1}, \eta_k)=(\eta_L, \eta_R)$ on the ζ-coordinate line $\zeta=\zeta_j$. Following the analogy of the steady problem (6) with the unsteady problem (8) shown previously, and the work in Reference 11, it is appropriate to construct the approximate Jacobian $\tilde{A}=\tilde{A}(w_L, w_R)$ in this cell such that,

$$\tilde{A}\,\Delta F=\Delta G, \tag{21}$$

for all jumps ΔF, where $\Delta(\cdot)=(\cdot)_R-(\cdot)_L$. This will mean that oblique jumps will be captured automatically. The matrix $\tilde{A}$ is assumed to have the form of $A=CB^{-1}$ with the flow variables ϕ, u and w replaced by averages $\tilde{\phi}$, $\tilde{u}$ and $\tilde{w}$, over the cell (η_L, η_R), and these averages are determined by solving (21). Moreover, we assume the grid derivatives x_η, z_η, x_ζ, z_ζ (and hence the Jacobian J) take constant (known) values X, Z, X' and Z', respectively, in this interval.

Equivalently, we could seek matrices $\tilde{B}$ and $\tilde{C}$ such that,

$$\tilde{B}\,\Delta w=\Delta F \quad \text{and } \tilde{C}\,\Delta w=\Delta G, \tag{22a–b}$$

for any jump Δw, and hence by combining (21) and (22a–b),

$$\tilde{A}=\tilde{C}\tilde{B}^{-1}, \tag{23}$$

where the form for $\tilde{B}$ and $\tilde{C}$ is assumed to be as in (15) and (16), where ϕ, u, w, x_η, z_η, x_ζ and z_ζ are replaced by $\tilde{\phi}$, $\tilde{u}$, $\tilde{w}$, X, Z, X' and Z', respectively.

Denoting the eigenvalues of $\tilde{A}$ by $\tilde{\lambda}_i$ with corresponding eigenvectors $\tilde{r}_i$ then similar relationships to those presented previously hold, i.e. $\tilde{\mathscr{A}}$ has eigenvalues $\tilde{\lambda}_i$ with eigenvectors $\tilde{e}_i$ where

$$(\tilde{C}-\tilde{\lambda}_i\tilde{B})\tilde{e}_i=0 \qquad i=1,2,3 \tag{24}$$

$$\tilde{B}\tilde{e}_i=\tilde{r}_i \qquad i=1,2,3. \tag{25}$$

Solving (24) gives,

$$\tilde{\lambda}_{1,2}=\frac{\tilde{U}\tilde{W}+\tilde{\phi}(XX'+ZZ')\pm\tilde{\phi}\sqrt{\tilde{d}}}{\tilde{U}^2-\tilde{\phi}(X^2+Z^2)} \tag{26c–b}$$

and

$$\tilde{\lambda}_3=\frac{\tilde{W}}{\tilde{U}} \tag{26c}$$

where

$$\tilde{U}=Z\tilde{u}-X\tilde{w}, \qquad \tilde{W}=-Z'\tilde{u}+X'\tilde{w} \tag{27a–b}$$

and

$$\tilde{d}=\frac{\tilde{U}^2(X'^2+Z'^2)+\tilde{W}^2(X^2+Z^2)}{\tilde{\phi}}-\tilde{J}^2 \tag{28}$$

with

$$\tilde{J}=X'Z-XZ' \tag{29}$$

and the averages $\tilde{\phi}$, $\tilde{u}$ and $\tilde{w}$ are still to be determined. The solution of (6) is then obtained from approximate solutions to the Riemann problem,

$$w_\zeta+\tilde{\mathscr{A}}w_\eta=0 \tag{30}$$

using upwind differences, where $\tilde{\mathscr{A}}=\tilde{B}^{-1}\tilde{C}$ (compare with (13)).

Averages

Using the form for $\tilde{A}$ (or $\tilde{B}$ and $\tilde{C}$) as noted above, and solving (21) (or (22a–b)), we obtain the following averages for $\tilde{\phi}$, $\tilde{u}$ and $\tilde{w}$ in (η_L, η_R)

$$\tilde{\phi}=\sqrt{\tfrac{1}{2}(\phi_L+\phi_R)} \tag{31a}$$

$$\tilde{u}=\frac{\sqrt{\phi_L}u_L+\sqrt{\phi_R}u_R}{\sqrt{\phi_L}+\sqrt{\phi_R}} \tag{31b}$$

and

$$\tilde{w}=\frac{\sqrt{\phi_L}w_L+\sqrt{\phi_R}w_R}{\sqrt{\phi_L}+\sqrt{\phi_R}} \tag{31c}$$

which are required in the numerical scheme.

GRID TRANSFORMATION

We assume that the grid transformation from physical (x, z) space to computational (ζ, η) space is either known analytically, or can be constructed numerically[13]. In the case where the mapping $x=x(\zeta,\eta)$, $z=z(\zeta,\eta)$ is known analytically, we take the approximation, X, to x_η as $X=x_\eta(\zeta_j, \eta_{k-1/2})$, and similarly the approximation Z, X' and Z', and hence the approximate grid Jacobian $\tilde{J}=X'Z-XZ'$. Otherwise, we set X to be the arithmetic mean of central difference approximation to x_η at (ζ_j, η_{k-1}) and (ζ_j, η_k), and similarly for Z, X', Z' and $\tilde{J}$.

For the special case of a channel with a symmetrical transition, as shown in *Figure 1*, only one half of the flow in the channel needs to be computed, and a transformation which maps the domain into a rectangular one is,

$$\zeta=x \tag{32a}$$

$$\eta=\frac{z}{b(x)} \tag{32b}$$

where $b(x)$ is the distance between the symmetry line and the upper boundary at distance x along the line. Symmetry boundary conditions apply along $\eta=0$, and reflecting boundary conditions apply along $\eta=1$. In this case $x=x(\zeta,\eta)=\zeta$ and $z=z(\zeta,\eta)=b(\zeta)\eta$, so that,

$$x_\zeta=1, \qquad x_\eta=0, \qquad z_\zeta=b'(\zeta)\eta, \qquad z_\eta=b(\zeta) \tag{33a–d}$$

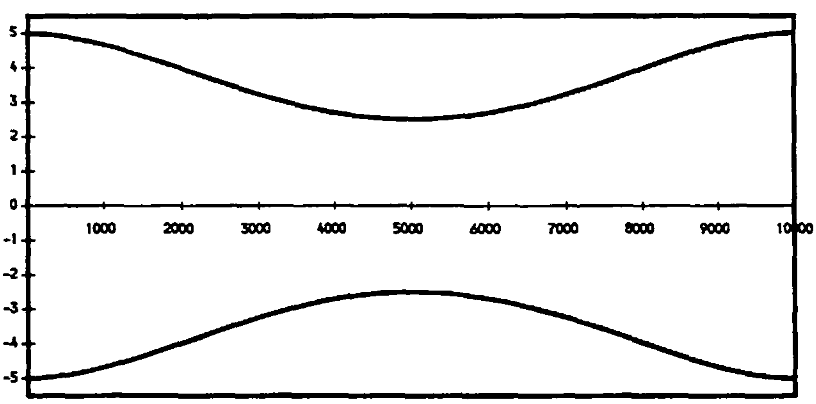

Figure 1 Channel breadth

and hence the corresponding approximations given above become,

$$X=0, \qquad Z=b(\zeta_j), \qquad X'=1, \qquad Z'=b'(\zeta_j)\eta_{k-1/2} \tag{34a–d}$$

and

$$\tilde{J}=b(\zeta_j) \tag{34e}$$

Thus, the approximate eigenvalues given by (26a–c) simplify to,

$$\tilde{\lambda}_{1,2}=\frac{\tilde{U}\tilde{W}+\tilde{\Phi}b(\zeta_j)b'(\zeta_j)\eta_{k-1/2}\pm\tilde{\Phi}\sqrt{\tilde{d}}}{\tilde{U}^2-\tilde{\Phi}b(\zeta_j)^2} \tag{35a–b}$$

$$\tilde{\lambda}_3=\frac{\tilde{W}}{\tilde{U}}, \tag{35c}$$

where

$$\tilde{U}=b(\zeta_j)\tilde{u}, \qquad \tilde{W}=-b(\zeta_j)\eta_{k-1/2}\tilde{u}+\tilde{w} \tag{36a–b}$$

and

$$\tilde{d}=\frac{\tilde{U}^2(1+b'(\zeta_j)^2\eta_{k-1/2}^2)+\tilde{W}^2b(\zeta_j)^2}{\tilde{\Phi}}-b(\zeta_j)^2 \tag{37}$$

NON-ZERO FRICTION AND BOTTOM SLOPE

In the case where the source term $f\neq 0$, we project an approximation to f evaluated at $(\zeta_j, \eta_{k-1/2})$ on to the eigenvectors e_i satisfying (24) and upwind this term in the same way that $\mathscr{A}w_\eta$ is treated in (30).

TEST PROBLEM AND NUMERICAL RESULTS

We present results for a test problem to verify the jump capturing capability of the scheme.

Consider the flow in a channel with a smooth constriction and a sloping bottom surface, which shelves. This geometry induces a flow which contains standing bores. The channel is

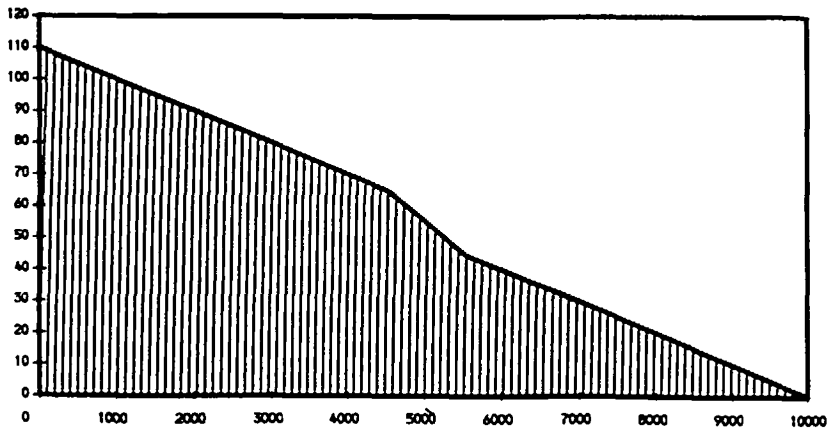

Figure 2 Channel cross-section

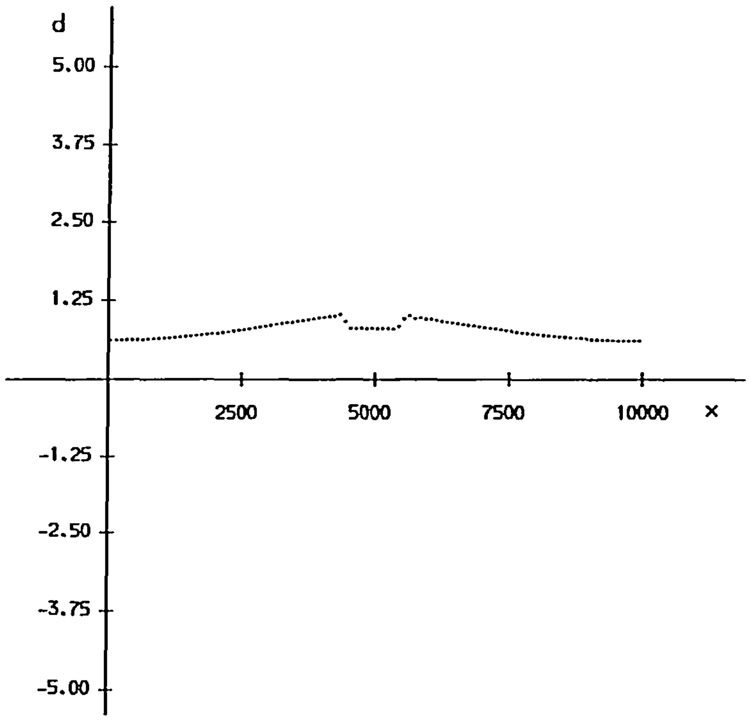

Figure 3 A plot of depth for a bed-slope of 0.02

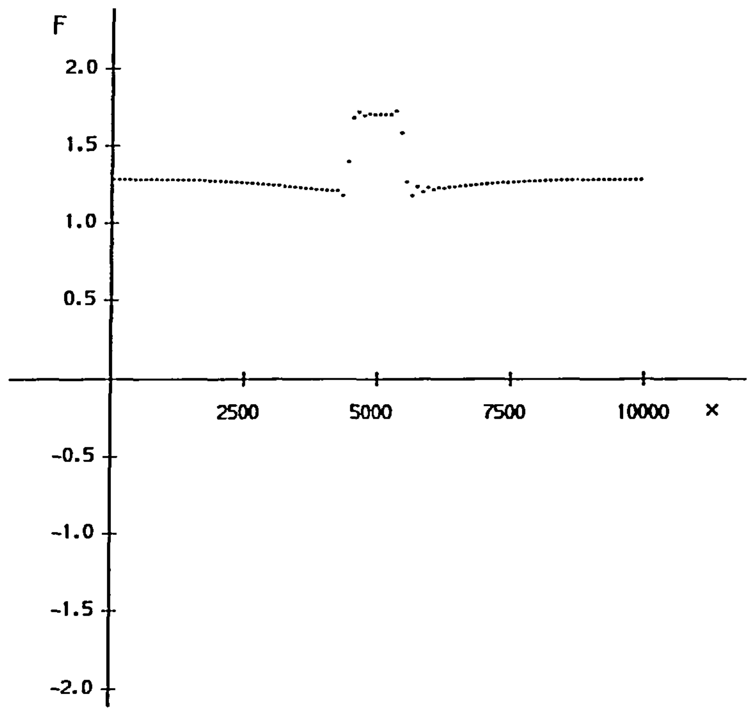

Figure 4 A plot of Froude number for a bed-slope of 0.02

10,000 metres long, and the breadth, B, varies from 10 metres to 5 metres to 10 metres (see *Figure 1*). The bed slope is taken to be a constant value, except between 4500 and 5500 metres where twice this value is taken (see *Figure 2*). Only one boundary condition needs to be applied to each end of the channel. At the left-hand end the mass flow is specified, and at the right-hand end the depth is specified by extrapolation from the interior.

Figures 3 and *4* show the results along the line of symmetry of the channel for a slope of 0.02, and the depth and the Froude number are displayed. The results are for 100 mesh points along the channel and 20 mesh points in the transverse direction and a mass flow of 20 cubic metres per second. The jumps are captured over two or three cells. These results compare well with those using other schemes, for example the algorithm in Reference 14.

CONCLUSIONS

We have presented a numerical scheme for the equations of steady, supercritical, open-channel flow. The scheme incorporates the use of boundary fitted coordinates, and the numerical results for a test problem demonstrate that the scheme can capture jumps over two or three cells. Importantly, the scheme is explicit and hence efficient in computational expense. The advantage of using boundary fitted coordinates when treating non-rectangular domains is well-known[13].

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